Mixtures of compound Poisson processes as models of tick-by-tick financial data

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Abstract

A model for the phenomenological description of tick-by-tick share prices in a stock exchange is introduced. It is based on mixtures of compound Poisson processes. Preliminary results based on Monte Carlo simulation show that this model can reproduce various stylized facts.

1 Introduction

Continuous time random walks (CTRWs) were introduced in Physics by Montroll and Weiss as a model for single-particle (tracer) diffusion [1]. An instance of CTRW, the normal compound Poisson process, had already been used in the probabilistic theory of insurance ruin since the beginning of the XXth Century [2,3].

The seminal paper of Montroll and Weiss has been followed by many studies focusing on anomalous relaxation and anomalous diffusion. This is the subject of two recent reviews by Metzler and Klafter [4,5].

The present author has recently reviewed the applications of CTRWs to Finance and Economics [6]. These applications were triggered by a series of papers on finance and fractional calculus [7,8,9], but the reader is referred to ref. [10] for an early application of the normal compound Poisson process to financial data.

The recent research of the present author has focused on the behaviour of waiting times (also known as durations) between trades and order in financial markets [8,11,12,13]. It turned out that interorder and intertrade waiting times are not exponentially distributed. Therefore, the jump process of tick-by-tick prices is non-Markovian [8].

In an article within this issue [14], Bianco and Grigolini apply a new method to verify whether the intertrade waiting time process is a genuine renewal process [15,16,17]. This was assumed by the CTRW hypothesis in [7]. They find that intertrade waiting times follow a renewal process.

Here, inspired by the work of Edelman and Gillespie [18,19], a phenomenological model for intraday tick-by-tick financial data is presented. It is able to reproduce some important stylized facts. The paper is organized as follows. Section 2 contains an outline of the theory of CTRWs. Section 3 contains a description of the model as well as a discussion on results from Monte Carlo simulations.

2   Outline of theory

2.1   Basic definitions

CTRWs are point processes with reward. The point process is characterized by a sequence of independent identically distributed (i.i.d.) positive random variables τ_i, which can be interpreted as waiting times between two consecutive events:

$$t_n = t_0 + \sum_{i=1}^{n} \tau_i; \;\; t_n - t_{n-1} = \tau_n; \;\; n = 1, 2, 3, \ldots; \;\; t_0 = 0. \qquad (1)$$

The rewards are i.i.d. not necessarily positive random variables: ξ_i. In the usual physical intepretation, they represent the jumps of a random walker, and they can be n-dimensional vectors. In this paper, only the 1-dimensional case is studied for a continuous random variable, but the extension of many results to the n-dimensional case and to a lattice is straightforward. The position x of the walker at time t is (with $N(t) = \max\{n : \; t_n \le t\}$ and $x(0) = 0$):

$$x(t) = \sum_{i=1}^{N(t)} \xi_i. \qquad (2)$$

CTRWs are rather good and general phenomenological models for diffusion, including anomalous diffusion, provided that the time of residence of the walker

is much greater than the time it takes to make a jump. In fact, in this formalism, jumps are instantaneous.

The financial interpretation of the random variables is straightforward. If trades take place in a continuous double auction, both price variations and waiting times (also called durations) between two consecutive trades are random variables. If $S(t)$ is the price of an asset at time t defined according to the previous tick interpolation procedure, $S(t) = S(t_i)$ where t_i is the time instant at which the last trade took place, then the price process can be considered as a pure jump stochastic process in continuous time. In finance, it is better to work with returns rather that prices. If $S(0)$ is the price at time $t = 0$, then the variable $x(t) = \log(S(t)/S(0))$ is called the *log-return* or, better, the *log-price*. This variable is analogous to the position of the walker in the physical interpretation. In the financial intepretation the jump random variables ξ_i are tick-by-tick log returns and they coincide with the difference between two consecutive log prices, whereas the waiting times or durations τ_i denote the elapsed time between two consecutive trades.

In general, jumps and waiting times are not independent from each other. In any case, a CTRW is characterized by the joint probability density $\varphi(\xi, \tau)$ of jumps and waiting times; $\varphi(\xi, \tau) \, d\xi \, d\tau$ is the probability of a jump to be in the interval $(\xi, \xi + d\xi)$ and of a waiting time to be in the interval $(\tau, \tau + d\tau)$. The following integral equation gives the probability density, $p(x, t)$, for the walker being in position x at time t, conditioned on the fact that it was in position $x = 0$ at time $t = 0$:

$$p(x,t) = \delta(x)\,\Psi(t) + \int_0^t \int_{-\infty}^{+\infty} \varphi(x - x', t - t')\, p(x', t')\, dt'\, dx', \qquad (3)$$

where $\Psi(\tau)$ is the so-called survival function. $\Psi(\tau)$ is related to the marginal waiting-time probability density $\psi(\tau)$. The two marginal densities $\psi(\tau)$ and $\lambda(\xi)$ are:

$$\psi(\tau) = \int_{-\infty}^{+\infty} \varphi(\xi, \tau)\, d\xi$$
$$\lambda(\xi) = \int_0^{\infty} \varphi(\xi, \tau)\, d\tau, \qquad (4)$$

and the survival function $\Psi(\tau)$ is defined as:

$$\Psi(\tau) = 1 - \int_0^{\tau} \psi(\tau')\, d\tau' = \int_{\tau}^{\infty} \psi(\tau')\, d\tau'. \qquad (5)$$

The integral equation, eq. (3) is linear and it can be solved in the Laplace-Fourier domain. The Laplace transform, $\widetilde{g}(s)$ of a (generalized) function $g(t)$

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is defined as:

$$\widetilde{g}(s) = \int_0^{+\infty} dt\, e^{-st}\, g(t)\,, \tag{6}$$

whereas the Fourier transform of a (generalized) function $f(x)$ is defined as:

$$\widehat{f}(\kappa) = \int_{-\infty}^{+\infty} dx\, e^{i\kappa x}\, f(x)\,. \tag{7}$$

A generalized function is a distribution (like Dirac's δ) in the sense of S. L. Sobolev and L. Schwartz [20].

One gets:

$$\widehat{\widetilde{p}}(\kappa, s) = \widetilde{\Psi}(s)\, \frac{1}{1 - \widehat{\widetilde{\varphi}}(\kappa, s)}\,, \tag{8}$$

or, in terms of the density $\psi(\tau)$:

$$\widehat{\widetilde{p}}(\kappa, s) = \frac{1 - \widetilde{\psi}(s)}{s}\, \frac{1}{1 - \widehat{\widetilde{\varphi}}(\kappa, s)}\,, \tag{9}$$

as, from eq. (5), one has:

$$\Psi(s) = \frac{1 - \widetilde{\psi}(s)}{s}\,. \tag{10}$$

In order to obtain $p(x, t)$, it is then necessary to invert its Laplace-Fourier transform $\widehat{\widetilde{p}}(\kappa, s)$. Analytic solutions are quite important, as they provide a benchmark for testing numerical inversion methods. In the next section, an explicit analytic solution for a class of continuous-time random walks with anomalous relaxation behaviour will be presented. It will be necessary to restrict oneself to the uncoupled case, in which jumps and waiting-times are not correlated.

2.2 The normal compound Poisson process

In this section, the solution of eq. (3) will be derived in the uncoupled case where the joint probability density of jumps and durations can be factorized in term of its marginals. After the derivation of a genearal formula for $p(x, t)$, this will be specialized to the case of the normal compound Poisson process (NCPP).

If jump sizes do not depend on waiting times, the joint probability density for jumps and waiting times can be written as follows:

$$\varphi(\xi, \tau) = \lambda(\xi)\psi(\tau) \tag{11}$$

with the normalization conditions $\int d\xi \lambda(\xi) = 1$ and $\int d\tau \psi(\tau) = 1$.

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In this case the integral master equation for $p(x,t)$ becomes:

$$p(x,t) = \delta(x)\,\Psi(t) + \int_0^t \psi(t-t') \left[\int_{-\infty}^{+\infty} \lambda(x-x')\,p(x',t')\,dx'\right] dt' \qquad (12)$$

This equation has a well known general explicit solution in terms of $P(n,t)$, the probability of n jumps occurring up to time t, and of the n-fold convolution of the jump density, $\lambda_n(x)$:

$$\lambda_n(x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} d\xi_{n-1}d\xi_{n-2}\ldots d\xi_1 \lambda(x-\xi_{n-1})\lambda(\xi_{n-1}-\xi_{n-2})\ldots\lambda(\xi_1). \qquad (13)$$

Indeed, $P(n,t)$ is given by:

$$P(n,t) = \int_0^t \psi_n(t-\tau)\Psi(\tau)\,d\tau \qquad (14)$$

where $\psi_n(\tau)$ is the n-fold convolution of the waiting-time density:

$$\psi_n(\tau) = \int_0^\tau \int_0^{\tau_{n-1}} \ldots \int_0^{\tau_1} d\tau_{n-1}d\tau_{n-2}\ldots d\tau_1 \psi(t-\tau_{n-1})\psi(\tau_{n-1}-\tau_{n-2})\ldots\psi(\tau_1). \qquad (15)$$

The n-fold convolutions defined above are probability density functions for the sum of n variables.

The Laplace transform of $P(n,t)$, $\widetilde{P}(n,s)$, reads:

$$\widetilde{P}(n,s) = [\widetilde{\psi}(s)]^n \widetilde{\Psi}(s) \qquad (16)$$

By taking the Fourier-Laplace transform of eq. (12), one gets:

$$\widetilde{\widehat{p}}(\kappa,s) = \widetilde{\Psi}(s)\,\frac{1}{1-\widetilde{\psi}(s)\widehat{\lambda}(\kappa)}\,. \qquad (17)$$

But, recalling that $|\lambda(\kappa)| < 1$ and $|\psi(s)| < 1$, if $\kappa \neq 0$ and $s \neq 0$, eq. (17) becomes:

$$\widetilde{\widehat{p}}(\kappa,s) = \widetilde{\Psi}(s)\sum_{n=0}^{\infty}[\widetilde{\psi}(s)\widehat{\lambda}(\kappa)]^n\,; \qquad (18)$$

this gives, inverting the Fourier and the Laplace transforms and taking into account eqs. (13) and (14):

$$p(x,t) = \sum_{n=0}^{\infty} P(n,t)\lambda_n(x) \qquad (19)$$

Eq. (19) can also be used as the starting point to derive eq. (12) via the transforms of Fourier and Laplace, as it describes a jump process subordinated to a renewal process.

A remarkable analytic solution is available when the waiting-time probability density function has the following exponential form:

$$\psi(\tau) = \mu e^{-\mu\tau}. \tag{20}$$

Then, the survival probability is $\Psi(\tau) = e^{-\mu\tau}$ and the probability of n jumps occurring up to time t is given by the Poisson distribution:

$$P(n,t) = \frac{(\mu t)^n}{n!} e^{-\mu t}. \tag{21}$$

This is the only Markovian case, and equation (19) becomes:

$$p(x,t) = \sum_{n=0}^{\infty} \frac{(\mu t)^n}{n!} e^{-\mu t} \lambda_n(x). \tag{22}$$

If $\lambda(\xi)$ follows the normal distribution $N(\xi; \bar{\xi}, \sigma_\xi)$, then the n-fold convolution is given by: $\lambda_n(x) = N(x; n\bar{\xi}, \sqrt{n}\sigma_\xi)$.

Given a series of empirical tick-by-tick log returns, $\{\xi_i\}_i^M$, as well as durations, $\{\tau_i\}_i^M$, one can directly evaluate the three parameters μ: the activity of the Poisson process, $\bar{\xi}$: the average of log-returns, and σ_ξ: the standard deviation of log-returns by means of suitable estimators [10].

However, the normal compound Poisson process is not able to reproduce the following stylized facts on high frequency data:

(1) The empirical distribution of log-returns is leptokurtic, whereas the NCPP assumes a mesokurtic (actually normal) distribution.
(2) The empirical distribution of durations is non-exponential with excess standard deviation [21,22,8,11,12], whereas the NCPP assumes an exponential distribution.
(3) The autocorrelation of absolute log-returns decays slowly [11], whereas the NCPP assumes i.i.d. log-returns.
(4) Log-returns and waiting times are not independent [11,23], whereas the NCPP assumes their independence.
(5) Volatility and activity vary during the trading day [24], whereas the NCPP assumes they are constant.

3 Mixtures of normal compound Poisson processes

3.1 Definition

It is possible to overcome the above shortcomings by using a suitable mixture of NCPPs. During a trading day, the volatility and the activity are higher at the opening of the market, then they decrease at midday and they increase again at market closure [24]. If the trading day can be divided into T intervals of constant activity $\{\mu_i\}_{i=1}^{T}$, then the waiting-time distribution is a mixture of exponential distributions and its probability density can be written as:

$$\psi(\tau) = \sum_{i=1}^{T} a_i \mu_i e^{-\mu_i \tau}, \tag{23}$$

where $\{a_i\}_{i=1}^{T}$ is a set of suitable weights. The activity seasonality can be mimicked by values of μ_i that decrease towards midday and then increase again towards market closure. In order to reproduce the correlation between volatility and activity, one can assume that:

$$\sigma_{\xi,i} = c\mu_i \tag{24}$$

where c is a suitable constant. Future work will be devoted to an analytical study of this model as well as to further empirical investigations on model validation. Below, the results of a simulation performed with the model are presented, and the performance of the model with respect to the stylized facts is discussed.

3.2 Results

A Monte Carlo simulation of the model described in the previous subsection has been performed by considering a trading day divided into ten intervals of constant activity with $\{\mu_i\}_{i=1}^{10} = 1/10, 1/20, 1/30, 1/40, 1/50, 1/40, 1/30, 1/20, 1/15, 1/10$ s^{-1}. For each value of μ_i, 100 exponentially distributed waiting times have extracted as well as 100 normally distributed log-returns with zero average and $\sigma_{\xi,i} = 0.001 \cdot \mu_i$. Therefore, there are 1000 values of waiting times and log returns in a trading day, representing a rather liquid share. The opening price is set to 100 arbitrary units (a.u.). In Fig. 1, a sample path is plotted for the price as a function of trading time. Fig. 2 and Fig. 3 represent the tick-by-tick time series of log-returns and waiting times respectively. For this particular simulation, the effect of variable activity and volatility can be detected by direct eye inspection.
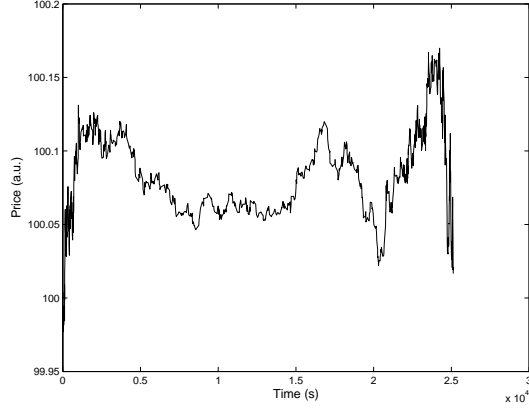
Fig. 1. Simulated price as a function of transaction time. The initial price is set to 100 arbitrary units (a.u.). Simulation times are measured in seconds
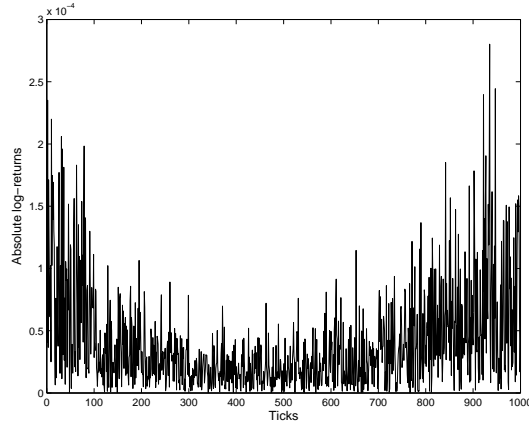

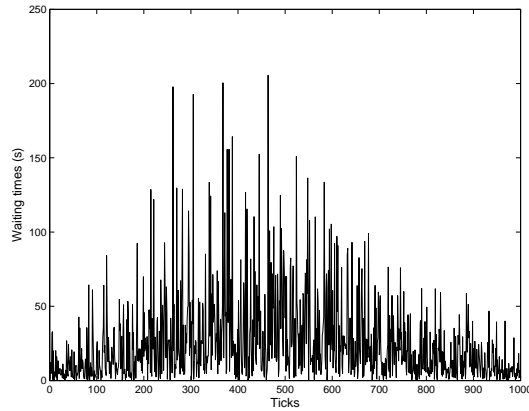Fig. 2. A simulated absolute log-return series.

Fig. 3. A simulated waiting time series.

In order to show that this model is able to reproduce the stylized facts described above, another set of figures is presented in the following. In Fig. 4 the empirical complementary cumulative distribution function is plotted for absolute tick-by-tick log returns. For comparison, the Gaussian fit with the
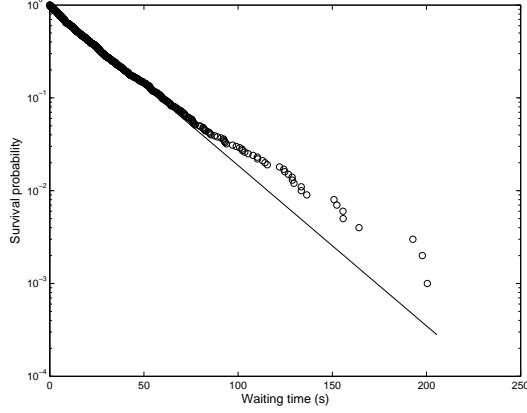
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Fig. 4. Empirical complementary cumulative distribution function for absolute log returns (circles). The solid line is a Gaussian fit.
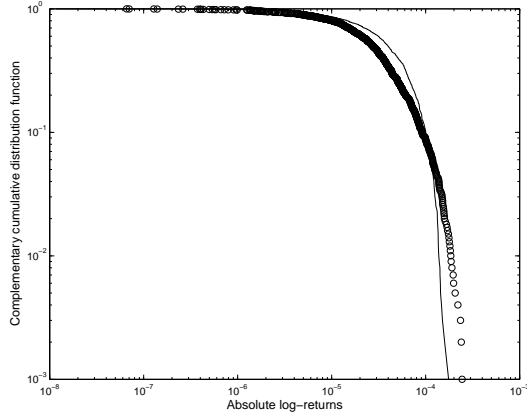

Fig. 5. Empirical survival probability.

same standard deviation of the 1000 log-returns is given by a solid line. This distribution has fat tails, is leptokurtic and the kurtosis is equal to 6.

The empirical complementary cumulative distribution function for intertrade durations is given in Fig. 5. The solid line is the single exponential fit to the simulated data. There is excess standard deviation: the standard deviation of waiting times is 29 s, whereas the average waiting time is 25 s.

Fig. 6 shows the slow decay of the autocorrelation of absolute log-returns related to volatility clustering, whereas signed log-returns are zero already at the second lag.

In conclusion, the model based on mixtures of normal compound Poisson processes incorporates variable daily activity, as well as the dependence between durations and tick-by-tick log-returns via eq. (24). It is then able to replicate the following stylized facts:

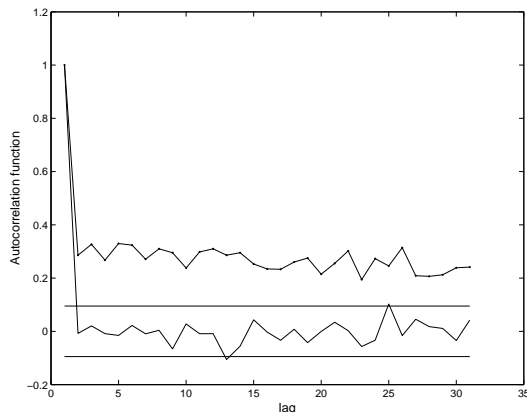- The empirical distribution of log-returns is leptokurtic;

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Fig. 6. Estimate of the autocorrelation function for absolute log-returns (.-), and signed log-returns (-). The solid horizontal lines represent the statistical zero level ($\pm 3/\sqrt{1000}$).

- the empirical distribution of durations is non-exponential with excess standard deviation;
- The autocorrelation of absolute log-returns decays slowly.

Work is currently in progress to empirically validate the model [25].

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